REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
11-07-2011	Final Performance Report	29-02-2008 to 11-07-2011

4. TITLE AND SUBTITLE

MULTISCALE COMPLEXITY IN LINEAR, DISPERSIVE PULSE PROPAGATION PHENOMENA

5a. CONTRACT NUMBER

5b. GRANT NUMBER

FA9550-08-1-0097

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Dr. Kurt E. Oughstun

5d. PROJECT NUMBER

07NE405

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

University of Vermont - School of Engineering
College of Engineering & Mathematical Sciences
Burlington, VT 05405

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

AFOSR / RSE / Dr. Nachman
875 North Randolph Street, Suite 325|
Room 3112
Arlington, Virginia 22203-1768

10. SPONSOR/MONITOR'S ACRONYM(S)

AFOSR / RSE

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

AFRL-OSR-VA-TR-2012-0225

12. DISTRIBUTION/AVAILABILITY STATEMENT

Distribution A

13. SUPPLEMENTARY NOTES

14. ABSTRACT

The classical asymptotic description of dispersive pulse propagation, initiated by Arnold Sommerfeld [1] and Leon Brillouin [2, 3] in 1914 using the then newly developed method of steepest descent due to Debye [4], and more recently completed by Oughstun & Sherman [5, 6, 7], Oughstun [8, 9], and Cartwright & Oughstun [10] using modern, uniform asymptotic expansion techniques, has provided an accurate mathematical description of ultrawideband pulse propagation in causally dispersive dielectrics and conducting media. The accuracy of this asymptotic description in- creases monotonically as the propagation distance z increases above some characteristic distance zd set by the material dispersion, typically given by the e−1 penetration depth at some oscillation frequency characteristic of the input pulse. This asymptotic description has also resulted in a simple physical description of dispersive pulse dynamics [11, 12] based upon the energy transport velocity [8, 13] and attenuation of a time-harmonic electromagnetic plane wave in the dispersive medium that reduces to the approximate group velocity description in a specific limit of vanishing loss (the limit in which the group velocity approximation is valid). What remains to be done in order to complete this

15. SUBJECT TERMS

Standard Terms Apply

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Dr Arje Nachman (Program Manager)
U	U	U	UU		19b. TELEPHONE NUMBER *(Include area code)* 703.696.8427

Reset

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

FINAL REPORT

AFOSR Grant # FA9550-08-1-0097

MULTI-SCALE COMPLEXITY IN LINEAR DISPERSIVE PULSE PROPAGATION PHENOMENA

Kurt E. Oughstun, Principle Investigator
School of Engineering
College of Engineering & Mathematical Sciences
University of Vermont

(February 2011)

1 Research Objectives

The classical asymptotic description of dispersive pulse propagation, initiated by Arnold Sommerfeld [1] and Leon Brillouin [2, 3] in 1914 using the then newly developed method of steepest descent due to Debye [4], and more recently completed by Oughstun & Sherman [5, 6, 7], Oughstun [8, 9], and Cartwright & Oughstun [10] using modern, uniform asymptotic expansion techniques, has provided an accurate mathematical description of ultrawideband pulse propagation in causally dispersive dielectrics and conducting media. The accuracy of this asymptotic description increases monotonically as the propagation distance z increases above some characteristic distance z_d set by the material dispersion, typically given by the e^{-1} penetration depth at some oscillation frequency characteristic of the input pulse. This asymptotic description has also resulted in a simple physical description of dispersive pulse dynamics [11, 12] based upon the energy transport velocity [8, 13] and attenuation of a time-harmonic electromagnetic plane wave in the dispersive medium that reduces to the approximate group velocity description in a specific limit of vanishing loss (the limit in which the group velocity approximation is valid). What remains to be done in order to complete this mathematically rigorous theory of dispersive pulse propagation is the development of a physical description of the precursor field formation at the molecular level, as it is there that the origin of the material dispersion occurs and the precursor field formation first appears. Associated with this problem is the correct physical description of dispersive pulse propagation in the immature dispersion regime where the precursor formation occurs. Although it has been asserted that the group velocity description would provide this near-field description, the research conducted in this grant has shown that this is is not necessarily true. These two related problems formed the focus of this research grant.

2 Research Results

A molecular theory of optics was originally developed by M. Born [14], L. Rosenfeld [15], and É. Lalor & E. Wolf [16]; a simpler asymptotic description of molecular optics may be found in the essential optics text [17] by M. Born & E. Wolf. This classical theory of molecular optics leads to the fundamentally important Ewald-Oseen extinction theorem which shows that a time-harmonic electromagnetic wave incident upon a dielectric interface initially penetrates into the medium undisturbed, exciting the molecules that comprise the medium which, in turn, produce a scattered wave field that extinguishes (through destructive interference) the incident plane wave field and constructs the refracted plane wave field. This analysis thus reveals the fundamental physical processes involved in the refraction of light at a dielectric interface. However, this analysis has only been completed for the special case of a time-harmonic electromagnetic wave. The extension of this theory of molecular optics to the time-dependent case was initiated in this grant with particular attention given to the ultrawideband signal, ultrashort pulse regime so as to reveal the fundamental physical processes involved in the precursor field formation over the initial penetration into the Lorentz-type dispersive half-space. Because the precursor fields are due to the intricate interplay between the phase delay dispersion $c/v_p(\omega) = n_r(\omega)$ and the attenuation dispersion $\alpha(\omega) = (\omega/c)n_i(\omega)$ of the dielectric medium, where the real $n_r(\omega) \equiv \Re\{n(\omega)\}$ and imaginary $n_i(\omega) \equiv \Im\{n(\omega)\}$ parts of the complex index of refraction $n(\omega) = [(\epsilon(\omega)/\epsilon_0)(\mu/\mu_0)]^{1/2}$ are interrelated through the appropriate dispersion relations required by causality [8, 18], they are an example of an emergent physical process and, as such, they present an important example of multi-scale complexity.

2.1 Integral Equation Representation of Electromagnetic Pulse Propagation in Dispersive Molecular Optics

Consider the propagation of an electromagnetic wave in a homogeneous, isotropic, nonmagnetic medium comprised of molecules that react to an incident field like ideal point dipoles. The electric and magnetic field vectors $\mathbf{E}'_j(\mathbf{r}, t)$ and $\mathbf{H}'_j(\mathbf{r}, t)$ which act on the j^{th} molecular dipole in the interior of the medium can then be formally separated into the superposition of the incident electromagnetic field vectors $\mathbf{E}^i(\mathbf{r}, t)$ and $\mathbf{H}^i(\mathbf{r}, t)$ that are propagating as if they were in vacuum with phase velocity c and the contribution arising from all of the other molecular dipoles in the medium,

so that

$$\mathbf{E}'_j(\mathbf{r},t) = \mathbf{E}^i(\mathbf{r},t) + \sum_\ell \mathbf{E}_{j\ell}(\mathbf{r},t), \tag{1}$$

$$\mathbf{H}'_j(\mathbf{r},t) = \mathbf{H}^i(\mathbf{r},t) + \sum_\ell \mathbf{H}_{j\ell}(\mathbf{r},t), \tag{2}$$

with $\ell \neq j$. At the point $\mathbf{r}_j$ where the j^{th} dipole is situated, the field of the ℓ^{th} dipole is given by the pair of expressions [17] (in cgs units)

$$\mathbf{E}_{j\ell}(\mathbf{r},t) = \nabla \times \nabla \times \frac{\mathbf{p}_\ell(t - R_{j\ell}/c)}{R_{j\ell}}, \tag{3}$$

$$\mathbf{H}_{j\ell}(\mathbf{r},t) = \frac{1}{c}\nabla \times \frac{\dot{\mathbf{p}}_\ell(t - R_{j\ell}/c)}{R_{j\ell}}, \tag{4}$$

where $\mathbf{p}_\ell(t)$ is the moment of the ℓ^{th} dipole, $R_{j\ell} \equiv |\mathbf{r}_j - \mathbf{r}_\ell|$, and where the spatial differentiation is taken with respect to the coordinates (x_j, y_j, z_j) of the j^{th} dipole.

The following simplifying, but not compromising, approximations are now made:

1. first, the spatial distribution of the molecular dipoles comprising the dielectric medium is treated as continuous so that $\mathbf{p}_\ell(t) \to \mathbf{p}(\mathbf{r},t)$;
2. and second, the number density $\mathcal{N}(\mathbf{r})$ of dipolar molecules is assumed to be a constant $\mathcal{N}$ in the material body and zero outside.

For a homogeneous, isotropic, locally linear dielectric exhibiting temporal dispersion, the total electric dipole moment per unit volume (*macroscopic polarization density*) is given by [8]

$$\mathbf{P}(\mathbf{r},t) = \mathcal{N}\int_{-\infty}^{t} \hat{\alpha}(t-t')\mathbf{E}'(\mathbf{r},t')dt', \tag{5}$$

where $\mathbf{E}'(\mathbf{r},t)$ is a spatial average of the effective local electric field intensity $\mathbf{E}'_j(\mathbf{r},t')$. For a causal medium response, Titchmarsh's theorem [8, 18] requires that $\hat{\alpha}(t-t') = 0$ for all $t' > t$, in which case the upper limit of integration may be extended to $+\infty$. The temporal Fourier transform of the resulting convolution relation then yields

$$\tilde{\mathbf{P}}(\mathbf{r},\omega) = \mathcal{N}\alpha(\omega)\tilde{\mathbf{E}}'(\mathbf{r},\omega), \tag{6}$$

where $\alpha(\omega) = \int_{-\infty}^{\infty} \hat{\alpha}(t)e^{i\omega t}dt$ is the mean molecular polarizability of the dielectric medium[1]. This molecular polarizability characterizes the frequency-dependent linear response of the molecules comprising the dielectric body to the applied electric field

[1]The term "mean" used here indicates a spatial average over molecular sites; see §4.4.1 of [8].

(see Appendix D of [8]). Notice that the 'simple' linear relation expressed in Eqs. (5) and (6) may be generalized to include spatial inhomogeneity by letting $\mathcal{N} \to \mathcal{N}(\mathbf{r})$ as well as anisotropy by generalizing the scalar quantity α to dyadic form. Spatial dispersion may also be included by removing the condition imposed by spatial locality in a generalized version of Eq. (5) by letting $\hat{\alpha}(t) \to \hat{\alpha}(\mathbf{r} - \mathbf{r}', t - t')$. For the current proposal, only a 'simple' linear, causally dispersive dielectric is considered.

Upon going over to a continuous distribution of the molecular dipoles in the dielectric medium, Eqs. (1)–(2) become, with substitution from Eqs. (3)–(4),

$$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}^i(\mathbf{r},t) + \int \nabla \times \nabla \times \left[\frac{\mathcal{N}}{R} \int_{-\infty}^{t} \hat{\alpha}(t-t') \mathbf{E}'(\mathbf{r}', t - R/c) dt' \right] d^3 r', \tag{7}$$

$$\mathbf{H}'(\mathbf{r},t) = \mathbf{H}^i(\mathbf{r},t) + \frac{1}{c} \int \nabla \times \left[\frac{\mathcal{N}}{R} \int_{-\infty}^{t} \hat{\alpha}(t-t') \dot{\mathbf{E}}'(\mathbf{r}', t - R/c) dt' \right] d^3 r', \tag{8}$$

where $R \equiv |\mathbf{r} - \mathbf{r}'|$, and where the indicated spatial differentiations are taken with respect to the unprimed coordinates (x, y, z) of the field point $\mathbf{r}$. If the point of observation (or field point) $\mathbf{r}$ is outside the dielectric medium, then the integration is taken throughout the entire space occupied by the medium. However, if it is inside the medium, then a small domain $\mathcal{D}_\Delta$ occupied by the molecule at that observation point must be excluded from the integration domain. As a matter of course, one then takes the limit as $\mathcal{D}_\Delta \to 0$ after the analysis has been completed.

Equation (7) is an integro-differential equation for the electric field vector $\mathbf{E}'(\mathbf{r}, t)$. When it is solved, the magnetic intensity vector $\mathbf{H}'(\mathbf{r}, t)$ associated with that electromagnetic wave field is then obtained from Eq. (8). This pair of relations is essentially equivalent to Maxwell's equations for homogeneous, isotropic, locally linear, nonmagnetic materials that exhibit temporal dispersion. Their solution has been obtained [14, 15, 16] for the special case when the incident wave field is strictly monochromatic with angular frequency ω, in which case the mean polarizability may be treated as a constant with $\hat{\alpha} = \alpha \delta(t - t')$.

When the incident electromagnetic wave is pulsed, and especially when it is ultrawideband with respect to the material dispersion, special care must be taken in solving Eqs. (7)–(8). These integro-differential equations are somewhat simplified in the temporal frequency domain obtained by taking the temporal Fourier-Laplace transform

of these relations, with the result

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \tilde{\mathbf{E}}^i(\mathbf{r},\omega) + \mathcal{N}\alpha(\omega)\int \nabla\times\nabla\times\left[\tilde{\mathbf{E}}'(\mathbf{r}',\omega)\frac{e^{i\omega R/c}}{R}\right]d^3r', \tag{9}$$

$$\tilde{\mathbf{H}}'(\mathbf{r},\omega) = \tilde{\mathbf{H}}^i(\mathbf{r},\omega) - \frac{i\omega}{c}\mathcal{N}\alpha(\omega)\int \nabla\times\left[\tilde{\mathbf{E}}'(\mathbf{r}',\omega)\frac{e^{i\omega R/c}}{R}\right]d^3r'. \tag{10}$$

Let Σ denote the closed boundary surface of the dielectric body occupying the region $\mathcal{D}\in\mathcal{R}^3$. For any observation point $\mathbf{r}\in\mathcal{D}$ inside the dielectric, Eqs. (6) and (9) may be combined to yield the relation

$$\tilde{\mathbf{P}}(\mathbf{r},\omega) = \mathcal{N}\alpha(\omega)\left[\tilde{\mathbf{E}}^i(\mathbf{r},\omega) + \tilde{\mathbf{E}}^d(\mathbf{r},\omega)\right], \tag{11}$$

where

$$\tilde{\mathbf{E}}^d(\mathbf{r},\omega) \equiv \int_{\mathcal{D}-\mathcal{D}_\Delta} \nabla\times\nabla\times\left[\tilde{\mathbf{P}}(\mathbf{r}',\omega)\frac{e^{i\omega R/c}}{R}\right]d^3r' \tag{12}$$

denotes the contribution from the molecular dipoles in the dielectric body $\mathcal{D}$ excluding the small region $\mathcal{D}_\Delta$ about the observation (or field) point $\mathbf{r}$. The electric field vector

$$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}^i(\mathbf{r},t) + \mathbf{E}^d(\mathbf{r},t) \tag{13}$$

acting on the molecular dipoles in the interior of the dielectric body may then be obtained from the inverse Fourier transform of the expression [cf. Eq. (6)]

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \frac{1}{\mathcal{N}\alpha(\omega)}\tilde{\mathbf{P}}(\mathbf{r},\omega). \tag{14}$$

The solution then depends upon the particular form of the macroscopic polarization density $\mathbf{P}(\mathbf{r},t)$. For example, for the nondispersive case the dipole moment density is taken to satisfy the homogeneous wave equation $\nabla^2\mathbf{P}(\mathbf{r},t) - (n^2/c^2)\partial^2\mathbf{P}(\mathbf{r},t)/\partial t^2 = \mathbf{0}$ where the constant n is an unknown quantity that is to be determined [16, 17].

For a Lorentz model dielectric, the macroscopic polarization density $\mathbf{P}(\mathbf{r},t)$ is derived form the microscopic equation of motion

$$m_e\left(\ddot{\mathbf{s}}_j + 2\delta\dot{\mathbf{s}}_j + \omega_0^2\mathbf{s}_j\right) = -q_e\mathbf{E}'_j(\mathbf{r},t), \tag{15}$$

where m_e is the mass of the electron and q_e the magnitude of the electronic charge, $\mathbf{E}'_j(\mathbf{r},t)$ being identified as the effective local electric field intensity acting on the j^{th} molecular dipole as driving force (compare the definition of the field vectors $\mathbf{E}'_j(\mathbf{r},t)$ and $\mathbf{H}'_j(\mathbf{r},t)$ in Eqs. (1) and (2) here with Eq. (4.200) of [8]). The additional force (in cgs units) $-q_e(\dot{\mathbf{s}}_j/c)\times\mathbf{B}'_j(\mathbf{r},t)$, arising from the interaction of the moving charge with

the effective local magnetic field, is assumed here to be negligible in comparison to the electric field interaction (due to the smallness of the magnitude of this charge velocity in comparison with the vacuum speed of light c), as described in [23]. Here ω_0 is the undamped resonance frequency, and δ is the associated phenomenological damping constant of the microscopic oscillator. The same dynamical equation of motion also applies to molecular vibration modes when m_e is replaced by the ionic mass and ω_0 is the undamped resonance frequency of the transverse vibrational mode of the ionic lattice structure [24].

The temporal frequency transform of Eq. (15) directly yields the frequency domain solution

$$\tilde{\mathbf{s}}_j(\mathbf{r},\omega) = \frac{q_e/m_e}{\omega^2 - \omega_0^2 + 2i\delta\omega}\tilde{\mathbf{E}}'_j(\mathbf{r},\omega), \tag{16}$$

so that the locally induced dipole moment $\tilde{\mathbf{p}}_j(\mathbf{r},\omega) = -q_e\tilde{\mathbf{s}}_j(\mathbf{r},\omega)$ of the j^{th} molecular dipole is given by

$$\tilde{\mathbf{p}}_j(\mathbf{r},\omega) = \alpha(\omega)\tilde{\mathbf{E}}'_j(\mathbf{r},\omega), \tag{17}$$

where

$$\alpha(\omega) = \frac{-q_e^2/m_e}{\omega^2 - \omega_0^2 + 2i\delta\omega} \tag{18}$$

is the mean molecular polarizability. The *macroscopic polarization density* is then given by the spatial average[2] of the locally induced (microscopic) dipole moments as

$$\tilde{\mathbf{P}}(\mathbf{r},\omega) = \mathcal{N}\langle\langle\tilde{\mathbf{p}}_j(\mathbf{r},\omega)\rangle\rangle = \mathcal{N}\alpha(\omega)\langle\langle\tilde{\mathbf{E}}'_j(\mathbf{r},\omega)\rangle\rangle. \tag{19}$$

With the identification that

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \langle\langle\tilde{\mathbf{E}}'_j(\mathbf{r},\omega)\rangle\rangle, \tag{20}$$

the expression in Eq. (19) reduces to that given in Eq. (6), viz.

$$\tilde{\mathbf{P}}(\mathbf{r},\omega) = \mathcal{N}\alpha(\omega)\tilde{\mathbf{E}}'(\mathbf{r},\omega). \tag{21}$$

Following the analytical tack taken by Lalor and Wolf in [16], it is now assumed that this polarization density satisfies the Helmholtz equation

$$\left(\nabla^2 + n^2(\omega)k_0^2\right)\tilde{\mathbf{P}}(\mathbf{r},\omega) = \mathbf{0}, \tag{22}$$

where $k_0 \equiv \omega/c$ denotes the wave number in vacuum, and where the complex-valued quantity $n(\omega)$ remains to be determined. As a trial solution, let [16]

$$\tilde{\mathbf{P}}(\mathbf{r},\omega) \equiv \left(n^2(\omega) - 1\right)k_0^2\tilde{\mathbf{Q}}(\mathbf{r},\omega). \tag{23}$$

[2]The definition of the spatial average of a given quantity $\star$, indicated by the double angle bracket notation $\langle\langle\star\rangle\rangle$, may be found in §4.1.1 of Ref. [8].

The spectral quantity $\tilde{\mathbf{Q}}(\mathbf{r},\omega)$ then satisfies the same vector Helmholtz equation

$$\left(\nabla^2 + n^2(\omega)k_0^2\right)\tilde{\mathbf{Q}}(\mathbf{r},\omega) = \mathbf{0} \tag{24}$$

as does the polarization density. The complex wave field $\tilde{\mathbf{Q}}(\mathbf{r},\omega)$ then "travels" in the dielectric body with the complex phase velocity $c/n(\omega)$. In addition, it is assumed that $\tilde{\mathbf{Q}}(\mathbf{r},\omega)$ has no sources[3] inside the medium, so that

$$\nabla \cdot \tilde{\mathbf{Q}}(\mathbf{r},\omega) = 0 \tag{25}$$

for all $\mathbf{r} \in \mathcal{D}$. Combination of Eqs. (11) and (23) then gives

$$\begin{aligned}\tilde{\mathbf{Q}}(\mathbf{r},\omega) &= \mathcal{N}\alpha(\omega)\Bigg\{\frac{1}{(n^2(\omega)-1)\,k_0^2}\tilde{\mathbf{E}}^i(\mathbf{r},\omega) \\ &\qquad + \int_{\mathcal{D}-\mathcal{D}_\Delta} \nabla\times\nabla\times\Big[\tilde{\mathbf{Q}})\mathbf{r}',\omega)G(R,\omega)\Big]\,d^3r'\Bigg\},\end{aligned} \tag{26}$$

where

$$G(R,\omega) \equiv \frac{e^{i\omega R/c}}{R} \tag{27}$$

is the *free-space Green's function* with $R = |\mathbf{r}-\mathbf{r}'|$. With the definition [16]

$$\tilde{\mathbf{A}}^d(\mathbf{r},\omega) \equiv \int_{\mathcal{D}-\mathcal{D}_\Delta} \nabla\times\nabla\times\Big[\tilde{\mathbf{Q}}(\mathbf{r}',\omega)G(R,\omega)\Big]\,d^3r', \tag{28}$$

the relation in Eq. (26) may be expressed in a more compact form as

$$\tilde{\mathbf{Q}}(\mathbf{r},\omega) = \mathcal{N}\alpha(\omega)\Bigg\{\frac{1}{(n^2(\omega)-1)\,k_0^2}\tilde{\mathbf{E}}^i(\mathbf{r},\omega) + \tilde{\mathbf{A}}^d(\mathbf{r},\omega)\Bigg\}. \tag{29}$$

Notice that the mathematical form of the spectral field quantity $\tilde{\mathbf{A}}^d(\mathbf{r},\omega)$ is precisely that obtained in the classical description of molecular optics [16], which was derived for a monochromatic wave field.

As the radius of the small spherical region $\mathcal{D}_\Delta$ surrounding the observation point $\mathbf{r}$ shrinks to zero, the integral appearing in Eq. (28) becomes (see Appendix V of Born and Wolf [17])

$$\int_{\mathcal{D}-\mathcal{D}_\Delta} \nabla\times\nabla\times\Big[\tilde{\mathbf{Q}}(\mathbf{r}',\omega)G(R,\omega)\Big]\,d^3r' = \nabla\times\nabla\times\int_{\mathcal{D}-\mathcal{D}_\Delta} \tilde{\mathbf{Q}}(\mathbf{r}',\omega)G(R,\omega)d^3r' - \frac{8\pi}{3}\tilde{\mathbf{Q}}(\mathbf{r},\omega). \tag{30}$$

[3]Note that this macroscopic polarization density is derived from the spatial average of the microscopic dipoles induced by the local effective electric field.

Because the free-space Green's function $G(R,\omega)$ describes a monochromatic spherical wave of angular frequency ω and wave number $k_0 = \omega/c$ in free-space, it then satisfies Helmholtz' equation

$$\left(\nabla^2 + k_0^2\right) G(R,\omega) = 0 \tag{31}$$

in the region $\mathcal{D} - \mathcal{D}_\Delta$. From Eqs. (24) and (31) one finds that

$$\tilde{\mathbf{Q}} G = \frac{1}{(n^2-1)k_0^2}\left(\tilde{\mathbf{Q}}\nabla^2 G - G\nabla^2\tilde{\mathbf{Q}}\right), \tag{32}$$

so that, with application of Green's theorem, the volume integral appearing on the right-hand side of Eq. (30) becomes

$$\begin{aligned}\int_{\mathcal{D}-\mathcal{D}_\Delta} \tilde{\mathbf{Q}}(\mathbf{r}',\omega) G(R,\omega) d^3r' &= \frac{1}{(n^2-1)k_0^2}\int_{\mathcal{D}-\mathcal{D}_\Delta}\left(\tilde{\mathbf{Q}}\nabla^2 G - G\nabla^2\tilde{\mathbf{Q}}\right) d^3r' \\ &= \frac{1}{(n^2-1)k_0^2}\left\{\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'} - G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right) d^2r' \right. \\ &\qquad \left. - \oint_{\Delta}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial R} - G\frac{\partial\tilde{\mathbf{Q}}}{\partial R}\right) d^2r'\right\}, \end{aligned} \tag{33}$$

where $\partial/\partial n'$ denotes differentiation along the outward normal n' [not to be confused with the complex index of refraction $n = n(\omega)$] to the boundary surface of the region $\mathcal{D} - \mathcal{D}_\Delta$. Notice that a minus sign appears in the second surface integral of Eq. (33) because $\partial/\partial R$ is the inner directed normal derivative to the spherical surface $\mathcal{D}_\Delta$. The evaluation of the limit of this inner surface integral over the spherical region $\mathcal{D}_\Delta$ as its radius R approaches zero closely follows that given in the derivation of the *integral theorem of Helmholtz and Kirchhoff* (see, for example, §8.3.1 of Born and Wolf [17]) with the result

$$\oint_{\Delta}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial R} - G\frac{\partial\tilde{\mathbf{Q}}}{\partial R}\right) d^2r' = -4\pi\tilde{\mathbf{Q}}(\mathbf{r}). \tag{34}$$

With this substitution, Eq. (33) becomes

$$\int_{\mathcal{D}} \tilde{\mathbf{Q}}(\mathbf{r}',\omega) G(R,\omega) d^3r' = \frac{1}{(n^2-1)k_0^2}\left\{\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'} - G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right) d^2r' + 4\pi\tilde{\mathbf{Q}}(\mathbf{r})\right\}. \tag{35}$$

Substitution of Eqs. (30) and (35) into Eq. (28) then gives

$$\begin{aligned}\tilde{\mathbf{A}}^d(\mathbf{r},\omega) &= \frac{1}{(n^2-1)\,k_0^2}\left\{4\pi\nabla\times\nabla\times\tilde{\mathbf{Q}}(\mathbf{r},\omega) + \nabla\times\nabla\times\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'} - G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right) d^2r'\right\} \\ &\qquad -\frac{8\pi}{3}\tilde{\mathbf{Q}}(\mathbf{r},\omega). \end{aligned} \tag{36}$$

Because $\tilde{\mathbf{Q}}(\mathbf{r},\omega)$ is solenoidal [see Eq. (25)], then, together with Eq. (24) it is found that

$$\nabla\times\nabla\times\tilde{\mathbf{Q}}(\mathbf{r},\omega)=\nabla\left(\nabla\cdot\tilde{\mathbf{Q}}(\mathbf{r},\omega)\right)-\nabla^2\tilde{\mathbf{Q}}(\mathbf{r},\omega)=n^2(\omega)k_0^2\tilde{\mathbf{Q}}(\mathbf{r},\omega), \tag{37}$$

and Eq. (36) becomes

$$\tilde{\mathbf{A}}^d(\mathbf{r},\omega)=\frac{4\pi}{3}\left(\frac{n^2+2}{n^2-1}\right)\tilde{\mathbf{Q}}(\mathbf{r},\omega)+\frac{1}{(n^2-1)\,k_0^2}\nabla\times\nabla\times\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'}-G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right)d^2r'. \tag{38}$$

Substitution of this expression into Eq. (29) then gives

$$\left[1-\frac{4\pi}{3}\left(\frac{n^2(\omega)+2}{n^2(\omega)-1}\right)\mathcal{N}\alpha(\omega)\right]\tilde{\mathbf{Q}}(\mathbf{r},\omega)$$
$$=\frac{\mathcal{N}\alpha(\omega)}{(n^2(\omega)-1)\,k_0^2}\left\{\tilde{\mathbf{E}}^i(\mathbf{r},\omega)+\nabla\times\nabla\times\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'}-G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right)d^2r'\right\}. \tag{39}$$

Because the left-hand side of this equation describes a monochromatic electromagnetic wave with complex phase velocity $c/n(\omega)$, whereas the right-hand side describes a wave that propagates with the velocity c, each side must then separately vanish. Thus

$$\boxed{\frac{4\pi}{3}\mathcal{N}\alpha(\omega)=\frac{n^2(\omega)-1}{n^2(\omega)+2}} \tag{40}$$

which is known either as the *Lorentz-Lorenz relation* [25, 26] or the *Clausius-Mossotti relation* [27, 28], and

$$\boxed{\tilde{\mathbf{E}}^i(\mathbf{r},\omega)+\nabla\times\nabla\times\oint_{\mathcal{S}}\left(\tilde{\mathbf{Q}}\frac{\partial G}{\partial n'}-G\frac{\partial\tilde{\mathbf{Q}}}{\partial n'}\right)d^2r'=\mathbf{0}} \tag{41}$$

which is known as the *Ewald-Oseen extinction theorem* [29, 30]. Because the Lorentz-Lorenz relation (40) may also be derived from a separate line of analysis (see, for example, §4.4.1 of [8]), the Ewalf-Oseen extinction theorem may also be viewed as a consequence of it as applied to Eq. (39).

The relation given in Eq. (41) expresses the *extinction of the incident spectral wave field component* $\tilde{\mathbf{E}}^i(\mathbf{r},\omega)$ at any point within the dielectric body through destructive interference with part of the induced dipole wave field. Notice that Eq. (41) states that the extinction of this incident wave field is brought about entirely by the dipoles on the boundary surface of the dielectric body.[4] The extinguished incident spectral

[4]This result is entirely consistent with the boundary conditions for the electric and magnetic field vectors at an interface $\mathcal{S}$ that are derived from the macroscopic Maxwell's field equations.

wave field at the frequency ω is then replaced by the spectral wave field at the same frequency ω that is given by [see Eqs. (13) and (11)]

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \tilde{\mathbf{E}}^i(\mathbf{r},\omega) + \tilde{\mathbf{E}}^d(\mathbf{r},\omega) = \frac{1}{\mathcal{N}\alpha(\omega)}\tilde{\mathbf{P}}(\mathbf{r},\omega). \tag{42}$$

With substitution from Eq. (23) and the Lorentz-Lorenz relation (40), this frequency-domain expression for the electric field vector acting on the molecular dipoles in the dielectric body becomes

$$\begin{aligned}\tilde{\mathbf{E}}'(\mathbf{r},\omega) &= \frac{1}{\mathcal{N}\alpha(\omega)}\left(n^2(\omega)-1\right)k_0^2\tilde{\mathbf{Q}}(\mathbf{r},\omega) \\ &= \frac{4\pi}{3}\left(n^2(\omega)+2\right)\frac{\omega^2}{c^2}\tilde{\mathbf{Q}}(\mathbf{r},\omega).\end{aligned} \tag{43}$$

which "propagates" inside the dielectric body with the *complex phase velocity*

$$v_p \equiv \frac{c}{n(\omega)}. \tag{44}$$

In terms of the macroscopic polarization vector $\tilde{\mathbf{P}}(\mathbf{r},\omega)$, this expression becomes

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \frac{4\pi}{3}\frac{n^2(\omega)+2}{n^2(\omega)-1}\tilde{\mathbf{P}}(\mathbf{r},\omega), \tag{45}$$

where $\tilde{\mathbf{P}}(\mathbf{r},\omega)$ satisfies the Helmholtz equation (22).

Finally, because $\tilde{\mathbf{D}}(\mathbf{r},\omega) = n^2(\omega)\tilde{\mathbf{E}}(\mathbf{r},\omega)$ in a homogeneous, isotropic, locally linear, nonmagnetic medium, and $\tilde{\mathbf{D}}(\mathbf{r},\omega) = \tilde{\mathbf{E}}(\mathbf{r},\omega) + 4\pi\tilde{\mathbf{P}}(\mathbf{r},\omega)$ in a simple polarizable dielectric, then the elimination of the macroscopic electric displacement vector spectrum from these two expressions yields

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \frac{4\pi}{n^2(\omega)-1}\tilde{\mathbf{P}}(\mathbf{r},\omega). \tag{46}$$

With Eq. (45) rewritten as

$$\tilde{\mathbf{E}}'(\mathbf{r},\omega) = \frac{4\pi}{n^2(\omega)-1}\tilde{\mathbf{P}}(\mathbf{r},\omega) + \frac{4\pi}{3}\tilde{\mathbf{P}}(\mathbf{r},\omega), \tag{47}$$

comparison with Eq. (46) then shows that the effective field acting on the molecular dipoles in the dielectric body is given by the well-known expression

$$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) + \frac{4\pi}{3}\mathbf{P}(\mathbf{r},t) \tag{48}$$

after a straightforward Fourier inversion into the space-time domain.

2.2 Precursor Field Formation in the Immature Dispersion Regime

Based upon the analytical formulation of dispersive molecular optics presented in §2.1, the time-domain development of the precursor fields due to an incident ultrawideband plane wave pulse may now be carefully investigated. Associated with this problem is the correct physical description of dispersive pulse propagation in the immature dispersion regime where the precursor formation occurs. Although it is generally asserted that the group velocity description provides this near-field description, the research conducted during this grant period has shown that this is not in general true.

2.2.1 Molecular Optics Formulation

Consider a linearly polarized plane wave pulse that is normally incident upon a dispersive dielectric filling the positive half space $z > 0$. Let the scalar field behavior of this incident pulse be described by $E^i(0^-, t) = E_0 f(t)$ where E_0 is a constant and $f(t)$ describes the temporal structure of the pulsed wave field. The macroscopic electric field intensity at any penetration distance $z \geq 0$ inside the dielectric half-space in then given by

$$\tilde{E}(z,\omega) = \frac{2}{1+n(\omega)} E_0 \tilde{f}(\omega) e^{i\tilde{k}(\omega)z}, \tag{49}$$

where the factor $2/(1+n(\omega))$ is the Fresnel transmission coefficient [17] for normal incidence,

$$\tilde{k}(\omega) \equiv \frac{c}{\omega} n(\omega) \tag{50}$$

is the complex wave number in the dispersive dielectric at the frequency ω, and where

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{51}$$

is the Fourier-Laplace spectrum [7, 8] of the initial pulse waveform. The spectrum of the induced macroscopic polarization is then obtained from Eq. (46) as

$$\tilde{P}(z,\omega) = \frac{1}{2\pi}\big(n(\omega)-1\big) E_0 \tilde{f}(\omega) e^{i\tilde{k}(\omega)z} \tag{52}$$

for all $z > 0$, and the spectrum of the electric field acting on the molecular dipoles in the interior of the dielectric half-space is obtained from Eq. (45) as

$$\tilde{E}'(z,\omega) = \frac{2}{3}\frac{n^2(\omega)+2}{n(\omega)+1} E_0 \tilde{f}(\omega) e^{i\tilde{k}(\omega)z} \tag{53}$$

for all $z > 0$. Finally, the temporal frequency spectrum of the dipole field in the dielectric half-space $z > 0$ is obtained from Eqs. (13) and (53) as

$$\tilde{E}^d(z,\omega) = \frac{2}{3}\frac{n^2(\omega)+2}{n(\omega)+1}E_0\tilde{f}(\omega)e^{i\tilde{k}(\omega)z} - E_0\tilde{f}(\omega)e^{ik_0 z} \tag{54}$$

where $k_0 = \omega/c$ is the wave number in vacuum. Because $\tilde{k}(\omega) - k_0 = \frac{\omega}{c}(n(\omega)-1)$, this expression can be written in more compact form as

$$\tilde{E}^d(z,\omega) = \left[\frac{2}{3}\frac{n^2(\omega)+2}{n(\omega)+1}e^{i\frac{\omega}{c}(n(\omega)-1)z} - 1\right]E_0\tilde{f}(\omega)e^{i\frac{\omega}{c}z}. \tag{55}$$

From Eqs. (42) and (48), the propagated plane wave field in the dispersive dielectric occupying the half-space $z > 0$ may be expressed as

$$E(z,t) = E^i(z,t) + E^d(z,t) - \frac{4\pi}{3}P(z,t) \tag{56}$$

This formulation will then reveal the origin of the precursor fields through the interaction of the dipole field $E^d(z,t)$ with the incident field $E^i(z,t)$ and the induced polarization field $P(z,t)$. This comprises a major part of my future research focus.

2.2.2 Energy and Group Velocity Descriptions

Both the energy and group velocity descriptions of dispersive pulse dynamics are based on (or derived from) the exact Fourier-Laplace integral representation of plane wave pulse propagation [1, 2, 7, 8]

$$A(z,t) = \frac{1}{2\pi}\int_C \tilde{f}(\omega)e^{\frac{z}{c}\phi(\omega,\theta)}d\omega, \tag{57}$$

where $A(0,t) = f(t)$ describes the initial pulse at $z = 0$ with spectrum

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \tag{58}$$

The *complex phase function* $\phi(\omega,\theta)$ appearing in Eq. (57) is given by

$$\phi(\omega,\theta) \equiv i\frac{c}{z}\left[\tilde{k}(\omega)z - \omega t\right] = i\omega[n(\omega) - \theta], \tag{59}$$

where

$$\tilde{k}(\omega) = \beta(\omega) + i\alpha(\omega) \equiv \frac{\omega}{c}n(\omega) \tag{60}$$

is the complex wavenumber of the electromagnetic wave in a dispersive nonmagnetic medium characterized by the complex index of refraction $n(\omega) = [\epsilon_c(\omega)/\epsilon_0]^{1/2}$, and where

$$\theta \equiv ct/z \tag{61}$$

is a *dimensionless space-time parameter* defined for all $z > 0$. If the complex index of refraction is described by a causal model, then Sommerfeld's relativistic causality theorem [1] applies, which states that [7, 9] *if* $A(0,t) = 0 \; \forall \; t < 0$, *then* $A(z,t) = 0 \; \forall \; z > 0$ *when* $ct/z < 1$, in agreement with the special theory of relativity. Physically meaningful examples of such causally dispersive medium models include the single relaxation time τ Debye-model dielectric

$$\epsilon(\omega)/\epsilon_0 = \epsilon_\infty + \frac{\epsilon_{sr} - \epsilon_\infty}{1 - i\omega\tau}, \tag{62}$$

with $\epsilon_\infty \geq 1$ and $\epsilon_{sr} \equiv \epsilon(0)/\epsilon_0$, the single resonance frequency ω_0 Lorentz-model dielectric

$$\epsilon(\omega)/\epsilon_0 = 1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega} \tag{63}$$

with $b^2 \equiv Nq_e^2/m_e$ the square of the plasma frequency with number density N and with damping constant $\delta \geq 0$, and the Drude model conductor

$$\epsilon_c(\omega)/\epsilon_0 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)} \tag{64}$$

with $\omega_p^2 \equiv Nq_e^2/m_e$ and with damping constant $\gamma = 1/\tau_c$ given by the inverse of the relaxation time τ_c associated with the mean-free for free electrons in the material.

With application of the saddle point method of analysis, the mathematical formulation given in Eqs. (57)–(60) leads to the uniform asymptotic representations [9] (for $\theta \geq 1$)

$$A(z,t) \sim A_b(z,t) + A_c(z,t) \tag{65}$$

as $z \to \infty$ for Debye-type dielectrics,

$$A(z,t) \sim A_s(z,t) + A_m(z,t) + A_b(z,t) + A_c(z,t) \tag{66}$$

as $z \to \infty$ for Lorentz-type dielectrics, and

$$A(z,t) \sim A_s(z,t) + A_b(z,t) + A_c(z,t) \tag{67}$$

as $z \to \infty$ for Drude-model conductors. Here $A_s(z,t)$ describes the first or Sommerfeld precursor wave whose dynamical evolution is described by the space-time evolution of the distant saddle points of $\phi(\omega,\theta)$ that appear in both Lorentz-model dielectrics and Drude-model conductors, $A_b(z,t)$ describes the second or Brillouin precursor wave whose dynamical evolution is described by the space-time evolution of the near

saddle points of $\phi(\omega, \theta)$ that appear in both Debye- and Lorentz model dielectrics as well as in Drude-model conductors, and $A_c(z,t)$ describes the signal contribution (if any). Uniform asymptotic descriptions of the Sommerfeld precursor have been given by Oughstun and Sherman [6, 7, 9] for Lorentz-model dielectrics and by Cartwright and Oughstun [31, 9] for Drude-model conductors. Uniform asymptotic descriptions of the Brillouin precursor have been given by Oughstun and Sherman [6, 7, 9] and by Cartwright and Oughstun [10] for Lorentz-model dielectrics, by Cartwright and Oughstun [31, 9] for Drude-model conductors, and by Oughstun [32] for Debye-model dielectrics. Finally, uniform asymptotic descriptions of the signal arrival and evolution have been given by Cartwright and Oughstun [10].

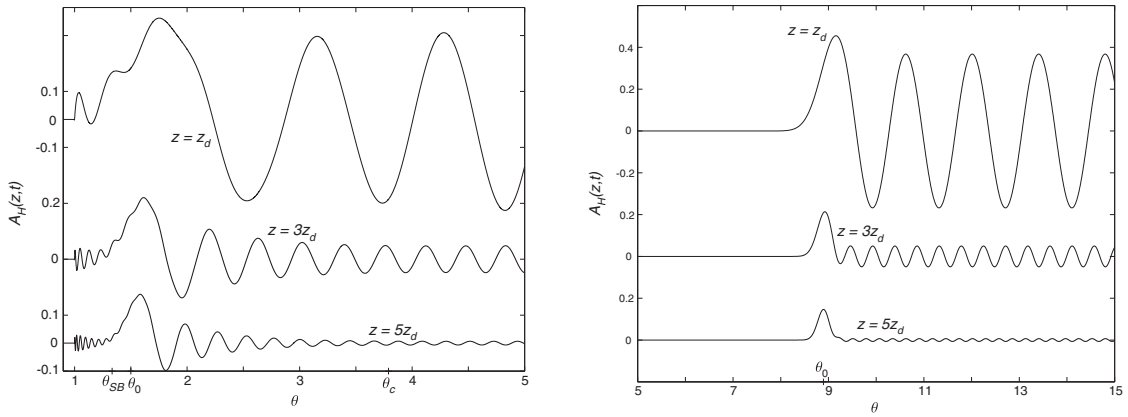

Figure 1. Dynamical evolution of a Heaviside step-function signal in (left) a Lorentz model dielectric (with below resonance carrier frequency $\omega_c < \omega_0$) and (right) a Debye model dielectric.

A comparison of the numerically determined field evolution due to a Heaviside unit step-function signal

$$A(0,t) = f_H(t) \equiv u_H(t)\sin(\omega_c t) \tag{68}$$

where $u_H(t) = 0$ for $t < 0$ and $u_H(t) = 1$ for $t > 0$ with constant carrier frequency ω_c is given in Fig. 1 for Lorentz- and Debye-model dielectrics at 1, 3, and 5 absorption depths z_d [where $z_d \equiv \alpha^{-1}(\omega_c)$] shows the difference between the two precursor field structures in these two dispersive medium types. The high-frequency Sommerfeld precursor evolution is clearly visible immediately following the propagated wave front (traveling at the vacuum speed of light c) in the Lorentz-model dielectric which is then followed by the low-frequency Brillouin precursor evolution and then the main signal evolution at the input signal carrier frequency ω_c, as described by Eq. (66) with $A_m(z,t) = 0$.[5] The observed asymmetry in both the Sommerfeld and Brillouin

[5]Because the propagation is in a single resonance Lorentz medium, the middle precursor field $A_m(z,t)$ is not present in the propagated field evolution.

precursor field evolutions is characteristic of the saddle point evolution in a Lorentz medium, as described in [5, 7, 9]. By comparison, the Sommerfeld precursor is completely absent from the dynamical field evolution in a Debye-model dielectric. In that case the propagated field evolution begins with the Brillouin precursor which is then followed by the main signal evolution at the input signal carrier frequency ω_c, as described by Eq. (65). Notice that the Brillouin precursor evolution is nearly symmetric about its peak amplitude point in this case, a characteristic of the saddle point evolution in a Debye medium, as described in [32, 8]. In both the Lorentz- and Debye-model cases, the peak amplitude of the Brillouin precursor only decays algebraically as $z^{-1/2}$ (except in the singular dispersion limit described below), this peak amplitude point occurring at the space-time point $\theta = \theta_0 \equiv n(0)$. The accuracy of the uniform asymptotic description in describing each feature in this type of dispersive pulse propagation phenomena is without peer.

One of the main criticism of this asymptotic theory is that it relies upon absorption and so cannot be applied to the optical domain where the materials (optical quality glass, etc.) are weakly dispersive and the group velocity description supposedly applies. The material absorption in a Lorentz medium (which is entirely appropriate for the description of the material dispersion in the optical region of the electromagnetic spectrum), described by the attenuation coefficient $\alpha(\omega) = (\omega/c)n_i(\omega)$ for real ω, decreases when either $\delta \to 0$ or when $N \to 0$. This then leads to the following two limiting cases:

- Singular Dispersion Limit: In the singular dispersion limit the frequency dispersion of $n(\omega)$ becomes increasingly localized about the resonance frequency ω_0 as $\delta \to 0$. The term "singular" is used here in the mathematical sense that $n(\omega)$ fails to be well-behaved at a point in some well-defined manner, in this case in terms of its differentiability at $\omega = \omega_0$. All of the material dispersion and loss is then concentrated at the resonance frequency.

- Weak Dispersion Limit: In the weak dispersion limit the absorption coefficient $\alpha(\omega)$ vanishes while the material dispersion $n_r(\omega)$ approaches unity at all frequencies ω as the molecular number density $N \to 0$. In this limit, $dn(\omega)/d\omega \to 0$ as $N \to 0$, in agreement with the group velocity interpretation of weak material dispersion.

The recent extension and verification of the asymptotic theory to both the singular and weak dispersion cases has shown that [22]:

- the peak amplitude decay of the Brillouin precursor changes from $z^{-1/2}$ to $z^{-1/3}$ in the singular dispersion limit as $\delta \to 0$, as illustrated in Fig. 2, and that

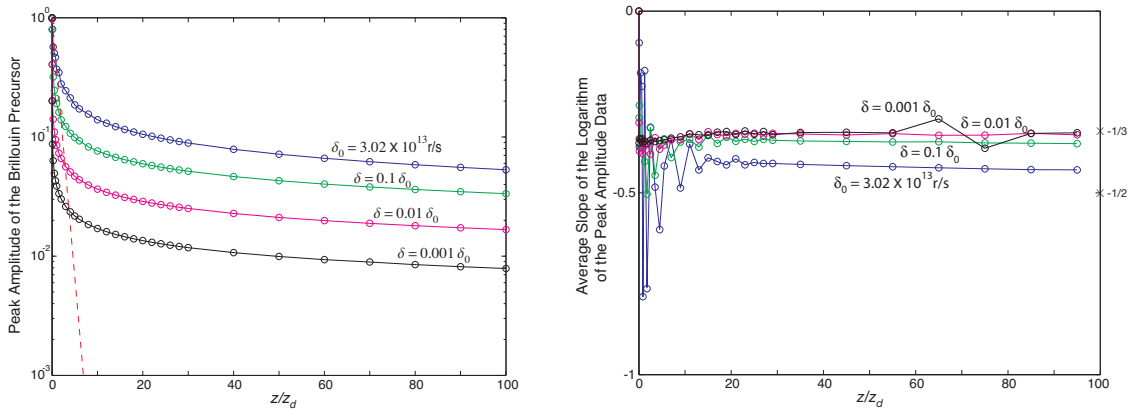

Figure 2. Numerically determined peak amplitude decay (left) and slope of the peak amplitude decay (right) in a Lorentz model dielectric in the singular dispersion limit as $\delta \to 0$.

- a simple scaling relation between strong and weak dispersion cases is given by

$$z_2 \approx \frac{N_1}{N_2} z_1 \quad \& \quad t_2 \approx t_1 + \left(\frac{N_1}{N_2} - 1\right) \frac{z_1}{c}, \tag{69}$$

  the accuracy of this result increasing as $N_1, N_2 \to 0$ with $N_2 > N_1$.

Most importantly, these results prove that the precursor fields persist in both the singular and weak dispersion limits where the group velocity description supposedly holds.

The energy velocity description [11, 12, 7, 9] directly follows from the asymptotic description and shows that the propagated wave field is dominated by a single real frequency ω_E at each space-time point $\theta = ct/z$ for all $z > 0$. That frequency ω_E is the angular frequency of the time-harmonic wave with the least attenuation that has energy velocity v_E equal to z/t, so that

$$v_E(\omega_E) = \frac{z}{t} \tag{70}$$

where [13]

$$v_E(\omega) = \frac{c}{n_r(\omega) + \omega n_i(\omega)/\delta} \tag{71}$$

for a single resonance Lorentz-model dielectric. For comparison, the group velocity in a Lorentz-model dielectric is given by

$$v_g(\omega) \equiv \frac{1}{\partial\beta/\partial\omega} = \frac{c}{n_r(\omega) + \omega n_r'(\omega)} \tag{72}$$

The question then arises regarding what conditions are required for $v_E(\omega) \to v_g(\omega)$, or equivalently, when does $n_i(\omega)/\delta \to n_r'(\omega)$?

For a single resonance Lorentz-model dielectric, the complex index of refraction $n(\omega) = n_r(\omega) + in_i(\omega)$ is given by

$$n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right)^{1/2} \tag{73}$$

with derivative

$$n'(\omega) = b^2 \frac{\omega + i\delta}{n(\omega)\left(\omega^2 - \omega_0^2 + 2i\delta\omega\right)^2}, \tag{74}$$

where $b^2 \equiv Nq_e^2/m_e$ is the square of the plasma frequency with number density N. In the singular dispersion limit as $\delta \to 0$,

$$n_r'(\omega) \to \frac{b^2\omega}{n_r(\omega)\left(\omega^2 - \omega_0^2\right)^2}, \tag{75}$$

$$\begin{aligned} n_i(\omega) &= \left[\frac{(\omega^2-\omega_0^2)^2 - b^2(\omega^2-\omega_0^2) + 4\delta^2\omega^2}{(\omega^2-\omega_0^2)^2 + 4\delta^2\omega^2}\right]^{1/2} \\ &\quad \times \frac{\delta b^2\omega}{(\omega^2-\omega_0^2)^2 - b^2(\omega^2-\omega_0^2) + 4\delta^2\omega^2}\left[1 + \mathcal{O}(\delta^2)\right] \\ &\to \delta \frac{b^2\omega}{n_r(\omega)(\omega^2-\omega_0^2)^2}, \end{aligned} \tag{76}$$

and the energy and group velocities are identical for all real ω. However, in the weak dispersion limit as $N \to 0$ (or equivalently as $b^2 \to 0$),

$$n(\omega) \to \underbrace{1 - \frac{(b^2/2)(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2 + 4\delta^2\omega^2}}_{n_r(\omega)} + i\,\underbrace{\delta \frac{b^2\omega}{(\omega^2-\omega_0^2)^2 + 4\delta^2\omega^2}}_{n_i(\omega)}, \tag{77}$$

$$n_r'(\omega) \to -\underbrace{\frac{b^2\omega}{(\omega^2-\omega_0^2)^2 + 4\delta^2\omega^2}}_{n_i(\omega)/\delta}\left[1 - 2\frac{1 + 2\delta^2/(\omega^2-\omega_0^2)}{1 + 4\delta^2\omega^2/(\omega^2-\omega_0^2)^2}\right] \tag{78}$$

and the energy and group velocities are approximately the same when either $\omega \ll \omega_0$ or $\omega \gg \omega_0$, the approximation improving as $\delta \to 0$.

This analysis then casts doubt on the applicability of the group velocity method to dispersive pulse propagation phenomena except in the singular dispersion limit. A proper description of dispersive pulse propagation then needs to be developed in the immature dispersion regime. Together with the closely related problem of the molecular theory of precursor field formation, this forms the major focus of my future research direction.

3 Peer-Reviewed Research Presentations & Publications

The following peer-reviewed conference presentations were given during this grant funding period:

1. N. A. Cartwright and K. E. Oughstun, "The Effect of Conductivity on the Brillouin Precursor," *2008 USNC/URSI National Radio Science Meeting*, University of Colorado, Boulder, CO (2008).

2. N. A. Cartwright and K. E. Oughstun, "The Effect of Conductivity on the Brillouin Precursor," *2008 Progress in Electromagnetics Research Symposium (PIERS 2008)*, Massachusetts Institute of Technology, Cambridge, MA (2008).

3. K. E. Oughstun, "Ultrawideband Pulse Propagation in Double-Resonance Lorentz Model Dielectrics," *2008 Progress in Electromagnetics Research Symposium (PIERS 2008)*, Massachusetts Institute of Technology, Cambridge, MA (2008).

4. N. A. Cartwright and K. E. Oughstun, "Pulse Propagation in a Debye Medium with Static Conductivity: The Search for a Uniform Expansion," *Special Session on Asymptotic Methods in Analysis with Applications, American Mathematical Society Annual Meeting*, Washington, D.C. (2009).

5. K. E. Oughstun, "On the Use & Application of Precursor Waveforms," *2009 13th International Symposium on Antenna Technology and Applied Electromagnetics and the Canadian Radio Science Meeting (ANTEM/URSI 2009)*, Banff Conference Centre, Banff, AB, Canada (2009).

6. K. E. Oughstun, "Dispersive Pulse Dynamics in the Few-Cycle Pulse Limit," (Invited Talk), *Nonlinear Optics in Guided Geometries*, Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany (2009).

7. K. E. Oughstun and N. A. Cartwright, "Brillouin Precursor Decay in Dispersive Attenuative Materials," *2009 IEEE International Symposium on Antennas and Propagation and USNC/URSI National Radio Science Meeting*, Charleston, SC (2009).

8. N. A. Cartwright and K. E. Oughstun, "Pulse Propagation in a Debye Medium with Static Conductivity," *2009 IEEE International Symposium on Antennas and Propagation and USNC/URSI National Radio Science Meeting*, Charleston, SC (2009).

9. K. E. Oughstun, "Utility of the Brillouin Precursor in Debye-Type Dielectrics," *2010 USNC/URSI National Radio Science Meeting*, University of Colorado, Boulder, CO (2010).

10. N. A. Cartwright and K. E. Oughstun, "Optical Precursors and the Group Velocity Approximation," *2010 IEEE AP-S International Symposium and USNC/CNC/URSI National Radio Science Meeting*, Toronto, Canada (2010).

11. K. E. Oughstun, "Beer's Law and the Unique Penetration Properties of the Brillouin Precursor in Complex Media," *2010 IEEE AP-S International Symposium and USNC/CNC/URSI National Radio Science Meeting*, Toronto, Canada (2010).

The following peer-reviewed conference publications appeared during this grant funding period:

1. K. E. Oughstun, "On the Use & Application of Precursor Waveforms," *Proceedings of the 2009 13th International Symposium on Antenna Technology and Applied Electromagnetics and the Canadian Radio Science Meeting (ANTEM/URSI 2009)*, pp. TP1:1–4 (2009).

2. K. E. Oughstun and N. A. Cartwright, "Brillouin Precursor Decay in Dispersive Attenuative Materials," *Proceedings of the 2009 IEEE International Symposium on Antennas and Propagation and USNC/URSI National Radio Science Meeting*, paper #528.5 (2009).

3. N. A. Cartwright and K. E. Oughstun, "Pulse Propagation in a Debye Medium with Static Conductivity," *Proceedings of the 2009 IEEE International Symposium on Antennas and Propagation and USNC/URSI National Radio Science Meeting*, paper #330.1 (2009).

4. K. E. Oughstun, "Beer's Law and the Unique Penetration Properties of the Brillouin Precursor in Complex Media," *Proceedings of the 2010 IEEE AP-S International Symposium and USNC/CNC/URSI National Radio Science Meeting*, paper #425.5 (2010).

The following peer-reviewed journal publications appeared during this grant funding period:

1. N. A. Cartwright and K. E. Oughstun, "Ultrawideband Pulse Propagation through a Homogeneous, Isotropic, Lossy Plasma," *Radio Science* **44**, RS4013–RS4024 (2009).

2. C. L. Palombini and K. E. Oughstun, "Optical Precursor Fields in Nonlinear Pulse Dynamics," *Optics Express* **18**, 22, 23104–23120 (2010).

3. K. E. Oughstun, N. A. Cartwright, D. J. Gauthier, and H. Jeong, "Optical Precursors in the Singular and Weak Dispersion Limits," *Journal of the Optical Society of America B* **27**, 8, 1664–1670 (2010).

4. K. E. Oughstun, N. A. Cartwright, D. J. Gauthier, and H. Jeong, "Optical Precursors in the Singular and Weak Dispersion Limits: Reply to Comment," *Journal of the Optical Society of America B* **28**, 3, 468–470 (2011).

In addition, the following research monograph was also published:

K. E. Oughstun, *Electromagnetic & Optical Pulse Propagation. Volume 2: Temporal Pulse Dynamics in Dispersive, Attenuative Media*, Springer Series in Optical Sciences (Springer, New York, 2009).

Finally, the following seminar and workshop presentations were given during this funding period:

1. "The Effect of Conductivity on the Brillouin Precursor," *19th Annual AFOSR Electromagnetics Workshop*, San Antonio, TX (January 8–10, 2008).

2. "On the Use & Application of Precursor Waveforms," *20th Annual AFOSR Electromagnetics Workshop*, San Antonio, TX (January 6–8, 2009).

3. "Dispersive Pulse Propagation: History, Theory, & Applications," *Engineering Seminar*, University of Toronto (February 12, 2009).

4. "Dispersive Pulse Propagation: History, Theory, & Applications," *Department of Physics Seminar*, Naval Postgraduate School, Monterey, CA (March 27, 2009).

5. "Precursor Field Behavior: Final Thoughts," *21st Annual AFOSR Electromagnetics Workshop*, San Antonio, TX (January 5–7, 2010).

6. "Precursor Wave Fields: The Characteristic Wave Structure of Dispersive Attenuative Media," *Institute of Optics Colloquium*, The Institute of Optics, The University of Rochester (April 12, 2010).

7. "Multus Tumultus de Nihilo," *22nd Annual AFOSR Electromagnetics Workshop*, San Antonio, TX (January 4–6, 2010).

The Latin phrase "Multus Tumultus de Nihilo" used as the title in this last workshop presentation was a reflection on the "Reply to Comment" paper listed above that formed the focus of this presentation.

References

[1] A. Sommerfeld, "Über die fortpflanzung des lichtes in disperdierenden medien," *Ann. Phys.* (Leipzig), **44**, 4, 177–202 (1914).

[2] L. Brillouin, "Über die fortpflanzung des licht in disperdierenden medien," *Ann. Phys.* (Leipzig), **44**, 4, 203–240 (1914).

[3] L. Brillouin, *Wave Propagation and Group Velocity*, (Academic Press, New York, 1960).

[4] P. Debye, "Näherungsformeln für die zylinderfunktionen für grosse werte des arguments und unbeschränkt verander liche werte des index," *Math. Ann.*, **67**, 535–558 (1909).

[5] K. E. Oughstun and G. C. Sherman, "Propagation of electromagnetic pulses in a linear dispersive medium with absorption (the Lorentz medium)," *J. Opt. Soc. Am. B* **5**, 4, 817–849 (1988).

[6] K. E. Oughstun and G. C. Sherman, "Uniform asymptotic description of electromagnetic pulse propagation in a linear dispersive medium with absorption (the Lorentz medium)," *J. Opt. Soc. Am. A* **6**, 9, 1394–1420 (1989).

[7] K. E. Oughstun and G. C. Sherman, *Electromagnetic Pulse Propagation in Causal Dielectrics*, (Springer-Verlag, Berlin-Heidelberg, 1994).

[8] K. E. Oughstun, *Electromagnetic & Optical Pulse Propagation 1: Spectral Representations in Temporally Dispersive Media*, (Springer, New York, 2006).

[9] K. E. Oughstun, *Electromagnetic & Optical Pulse Propagation 2: Temporal Pulse Dynamics in Dispersive Attenuative Media*, (Springer, New York, 2009).

[10] N. A. Cartwright and K. E. Oughstun, "Uniform asymptotics applied to ultrawideband pulse propagation," *SIAM Review* **49**, 4, 628–648 (2007).

[11] G. C. Sherman and K. E. Oughstun, "Description of Pulse Dynamics in Lorentz Media in Terms of the Energy Velocity and Attenuation of Time-Harmonic Waves," *Phys. Rev. Lett.* **47**, 20, 1451–1454 (1981).

[12] G. C. Sherman and K. E. Oughstun, "Energy Velocity Description of Pulse Propagation in Absorbing, Dispersive Dielectrics," *J. Opt. Soc. Am. B* **12**, 2, 229–247 (1995).

[13] R. Loudon, "The propagation of electromagnetic energy through an absorbing dielectric," *Phs. A* **3**, 233–245 (1970).

[14] M. Born, *Optik*, (Springer, Berlin, 1933), Ch. 7.

[15] L. Rosenfeld, *Theory of Electrons*, (North-Holland, Amsterdam, 1951).

[16] É. Lalor and E. Wolf, "Exact solution of the equation of molecular optics for refraction and reflection of an electromagnetic wave on a semi-infinite dielectric," *J. Opt. Soc. Am.* **62**, 10, 1165–1174 (1972).

[17] M. Born and E. Wolf, *Principles of Optics*, 7^{th} expanded edition (Cambridge University Press, Cambridge, 1999).

[18] H. M. Nussenzveig, *Causality and Dispersion Relations*, (Academic Press, New York, 1972), Ch. 1.

[19] J. Van Bladel, *Relativity and Engineering*, (Springer-Verlag, Berlin, 1984) §4.2.

[20] P. C. Clemmow, *The Plane Wave Spectrum Representation of Electromagnetic Fields*, (Pergamon, Oxford, 1966).

[21] J. J. Stamnes, *Radiation and Propagation of Light in Crystals*, (Ph.D. Thesis, The Institute of Optics, University of Rochester, Rochester, New York, 1974).

[22] K. E. Oughstun, N. A. Cartwright, D. J. Gauthier, and H. Jeong, "Optical precursors in the singular and weak dispersion limits," *J. Opt. Soc. Am. B* **27**, 8, 1664–1670 (2009).

[23] K. E. Oughstun and R. A. Albanese, "Magnetic field contribution to the Lorentz model," *J. Opt. Soc. Am. A* **23**, 7, 1751–1756 (2006).

[24] C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, (Wiley, New York, 1984), Ch. 9.

[25] H. A. Lorentz, "Über die Beziehungzwischen der Fortpflanzungsgeschwindigkeit des Lichtes der Körperdichte," *Ann. Phys.* **9**, 641–665 (1880).

[26] L. Lorenz, "Über die Refractionsconstante," *Ann. Phys.* **11**, 70–103 (1880).

[27] P. F. Mossotti, *Bibl. Univ. Modena*, vol. 6, p. 193 (1847).

[28] R. Clausius, *Die Mechanische Wärmetheorie*, vol. 2, pp. 62–97 (Braunschweich:Vieweg, 1879).

[29] P. P. Ewald, *Ann. Phys.* **49**, p. 1 (1916).

[30] C. W. Oseen, *Ann. Phys.* **48**, p.1 (1915).

[31] N. A. Cartwright and K. E. Oughstun, "Ultrawideband Pulse Propagation through a Homogeneous, Isotropic, Lossy Plasma," *Radio Science* **44**, RS4013–RS4024 (2009).

[32] K. E. Oughstun, "Dynamical Evolution of the Brillouin Precursor in Rocard-Powles-Debye Model Dielectrics," *IEEE Transactions on Antennas and Propagation* **53**, 5, 1582–1590 (2005).